

MAXIMAL REGULARITY FOR STOCHASTIC CONVOLUTIONS DRIVEN BY LEVY NOISE

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ABSTRACT. We show that the result from Da Prato and Lunardi is valid for stochastic convolutions driven by Lévy processes.

1. INTRODUCTION

The aim of the article is to investigate the maximal regularity of the Ornstein-Uhlenbeck driven by purely discontinuous noise. In particular, let (S, \mathcal{S}) be a measurable space, E be a Banach space of martingale type p , $1 < p \leq 2$, and A be an infinitesimal generator of an analytic semigroup $(e^{-tA})_{0 \leq t < \infty}$ in E . We consider the following SPDE written in the Itô-form

$$(1) \quad \begin{cases} du(t) &= Au(t-) dt + \int_S \xi(t; x) \tilde{\eta}(dx; dt), \\ u(0) &= 0, \end{cases}$$

where $\tilde{\eta}$ is a S -valued time homogeneous compensated Poisson random measure defined on a filtered probability space $(\Omega; \mathcal{F}; (\mathcal{F}_t)_{0 \leq t < \infty}; \mathbb{P})$ with Lévy measure ν on S , specified later, and $\xi : \Omega \times S \rightarrow E$ is a predictable process satisfying certain integrability conditions also specified later. The solution to (1) is given by the so called Ornstein-Uhlenbeck process

$$u(t) := \int_0^t \int_S e^{-A(t-r)} \xi(r, x) \tilde{\eta}(dx; dr), \quad t > 0.$$

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Suppose $1 \leq q \leq p$. Our main result will be the following inequality

$$(2) \quad \mathbb{E} \int_0^T |u(t)|_{D_A(\theta+\frac{1}{p}, q)}^p dt \leq C \mathbb{E} \int_0^T \int_S |\xi(t, z)|_{D_A(\theta, q)}^p dt,$$

where $D_A(\theta, p)$, $\theta \in (0, 1)$, denotes the real interpolation space of order θ between E and $D(A)$.

As mentioned in the beginning, if the Ornstein-Uhlenbeck process is driven by a scalar Wiener process, the question of maximal regularity was answered by Da Prato in [7] or Da Prato and Lunardi [8]. We transfer these results to the Ornstein-Uhlenbeck process driven by purely discontinuous noise.

Notation 1. By \mathbb{N} we denote the set of natural numbers, i.e. $\mathbb{N} = \{0, 1, 2, \dots\}$ and by $\bar{\mathbb{N}}$ we denote the set $\mathbb{N} \cup \{+\infty\}$. Whenever we speak about \mathbb{N} (or $\bar{\mathbb{N}}$)-valued measurable functions we implicitly assume that that set is equipped with the trivial σ -field $2^{\mathbb{N}}$ (or $2^{\bar{\mathbb{N}}}$). By \mathbb{R}_+ we will denote the interval $[0, \infty)$. If X is a topological space, then by $\mathcal{B}(X)$ we will denote the Borel σ -field on X . By λ we will denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For a measurable space (S, \mathcal{S}) let M_S^+ be the set of all non negative measures on (S, \mathcal{S}) .

2. MAIN RESULTS

Suppose that $p \in (1, 2]$ and that E is a Banach space of martingale type p . Let (S, \mathcal{S}) be a measurable space and $\nu \in M_S^+$. Suppose that $\mathfrak{P} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space, $\eta : \mathcal{S} \times \mathcal{B}(\mathbb{R}_+) \rightarrow \bar{\mathbb{N}}$ is time homogeneous Poisson random measure with intensity measure ν defined over $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to filtration $(\mathcal{F}_t)_{t \geq 0}$. We will denote by $\tilde{\eta} = \eta - \gamma$ the to η associated compensated Poisson random measure where γ is given by

$$\mathcal{B}(\mathbb{R}_+) \times \mathcal{S} \ni (A, I) \mapsto \gamma(A, I) = \nu(A)\lambda(I) \in \mathbb{R}_+.$$

We denote by \mathcal{P} the σ field on $\Omega \times \mathbb{R}_+$ generated by all sets $A \in \mathcal{F} \hat{\times} \mathcal{B}(\mathbb{R}_+)$, where A is of the form $A = F \times (s, t]$, with $F \in \mathcal{F}_s$ and $s, t \in \mathbb{R}_+$. If $\xi : \Omega \times \mathbb{R}_+ \rightarrow S$ is \mathcal{P} measurable, ξ is called predictable.

It is then known, see e.g. appendix B, that there exists a unique continuous linear operator associating with each predictable process $\xi : \mathbb{R}_+ \times S \times \Omega \rightarrow E$ with

$$(3) \quad \mathbb{E} \int_0^T \int_S |\xi(r, x)|^p \nu(dx) dr < \infty, \quad T > 0,$$

an adapted càdlàg process, denoted by $\int_0^t \int_S \xi(r, x) \tilde{\eta}(dx, dr)$, $t \geq 0$ such that if ξ satisfies the above condition (3) and is a step process

with representation

$$\xi(r) = \sum_{j=1}^n 1_{(t_{j-1}, t_j]}(r) \xi_j, \quad 0 \leq r,$$

where $\{t_0 = 0 < t_1 < \dots < t_n < \infty\}$ is a partition of $[0, \infty)$ and for all j , ξ_j is an $\mathcal{F}_{t_{j-1}}$ measurable random variable, then

$$(4) \quad \int_0^t \int_S \xi(r, x) \tilde{\eta}(dx, dr) = \sum_{j=1}^n \int_S \tilde{\xi}_j(x) \eta(dx, (t_{j-1} \wedge t, t_j \wedge t]).$$

The continuity mentioned above means that there exists a constant $C = C(E)$ independent of ξ such that

$$(5) \quad \mathbb{E} \left| \int_0^t \int_S \xi(r, x) \tilde{\eta}(dx, dr) \right|^p \leq C \mathbb{E} \int_0^t \int_S |\xi(r, x)|^p \nu(dx) dr, \quad t \geq 0.$$

One can prove¹, see e.g. the proof of Proposition 3.3 in [12], or Theorem 3.1 in [3] for the case $q < p$, and Corollary B.6 in Appendix B, that for any $q \in [1, p]$ there exists a constant $C = C_q(E)$ such that for each process ξ as above and for all $t \geq 0$,

$$(6) \quad \mathbb{E} \left| \int_0^t \int_S \xi(r, x) \tilde{\eta}(dx, dr) \right|^q \leq C \mathbb{E} \left(\int_0^t \int_S |\xi(r, x)|^p \nu(dx) dr \right)^{q/p}.$$

Remark 1. Let us denote

$$\begin{aligned} I(\xi)(t) &:= \int_0^t \int_S \xi(r, x) \tilde{\eta}(dx, dr), \quad t \geq 0 \\ \|\xi\| &:= \left(\int_S |\xi(x)|^p \nu(dx) \right)^{1/p}, \quad \xi \in L^p(S, \nu; E). \end{aligned}$$

Then the inequality (6) takes the following form

$$\mathbb{E} |I(\xi)(t)|^q \leq C_q(E) \mathbb{E} \left[\left(\int_0^t \|\xi(r)\|^p dr \right)^{q/p} \right].$$

This should be (and will be) compared with the Gaussian case. Note that in this case $\|\xi\|$ is simply the $L^p(S, \nu, E)$ norm of ξ . In the Gaussian case the situation is different.

Let us also point out that the inequality (6) for $q < p$ follows from the same inequality for $q = p$. In fact, using Proposition IV.4.7 from [20], see the proof of Theorem 3.1 in [3], one can prove a stronger result. Namely that if inequality (6) holds true for $q = p$, then for $q \in [1, p)$

¹The case $q \in (p, \infty)$ is different and will be discussed later.

there exists a constant $K_q > 0$ such that for each accessible stopping time $\tau > 0$,

$$(7) \quad \mathbb{E} \sup_{0 \leq t \leq \tau} |I(\xi)(t)|^q \leq K_q \mathbb{E} \left(\int_0^\tau \|\xi(t)\|^p dt \right)^{q/p}.$$

Assume further that $-A$ is an infinitesimal generator of an analytic semigroup denoted by $(e^{-tA})_{t \geq 0}$ on E .

Define the stochastic convolution of the semigroup $(e^{-tA})_{t \geq 0}$ and an E -valued process ξ as above by the following formula

$$(8) \quad SC(\xi)(t) = \int_0^t \int_S e^{(t-r)A} \xi(r, x) \tilde{\eta}(dx, dr), \quad t \geq 0.$$

Let us recall that there exist constants C_0 and ω_0 such that

$$\|e^{-tA}\| \leq C_0 e^{t\omega_0}, \quad t \geq 0.$$

Without loss of generality, we will assume from now on that $\omega_0 < 0$. Let us also recall the following characterization of the real interpolation² spaces $(E, D(A^m))_{\theta, q} = (D(A^m), E)_{1-\theta, q}$, where $m \in \mathbb{N}$, between $D(A^m)$ and E with parameters $\theta \in (0, 1)$ and $q \in [1, \infty)$, see section 1.14.5 in [21] or [7]. If $\delta \in (0, \infty]$ then

$$(D(A^m), E)_{1-\theta, q} = \left\{ x \in E : \int_0^\delta |t^{m(1-\theta)} A^m e^{-tA} x|^q \frac{dt}{t} < \infty \right\}.$$

$$(9) \quad (E, D(A^m))_{\vartheta, q} = \left\{ x \in E : \int_0^\delta |t^{m(1-\vartheta)} A^m e^{-tA} x|^q \frac{dt}{t} < \infty \right\}.$$

The norms defined by the equality (9) for different values of δ are equivalent.

The space $(D(A^m), E)_{1-\theta, q} = (E, D(A^m))_{\theta, q}$ is often denoted by $D_{A^m}(\theta, p)$ and we will use the following notation

$$(10) \quad |x|_{D_{A^m}(\theta, q); \delta}^q = \int_0^\delta |t^{m(1-\theta)} A^m e^{-tA} x|^q \frac{dt}{t}.$$

²In order to fix the notation let me point out that the interpolation functor $(X_0, X_1)_{\theta, q}$, $\theta \in (0, 1)$, $q \in [1, \infty]$, between two Banach spaces X_1 and X_0 such that both are continuously embedded into a common topological Hausdorff vector space, satisfies the following properties: (i) $(X_1, X_0)_{\theta, q} = (X_0, X_1)_{1-\theta, q}$, (ii) if $X_0 \subset X_1$, $0 < \theta_1 < \theta_2 < 1$ and $p, q \in [1, \infty]$, then $(X_0, X_1)_{\theta_1, p} \subset (X_0, X_1)_{\theta_2, q}$. Roughly speaking, (ii) implies that, if $X_0 \subset X_1$, then $(X_0, X_1)_{\vartheta, p} \searrow X_0$ as $\vartheta \searrow 0$ and $(X_0, X_1)_{\theta, p} \nearrow X_1$ as $\vartheta \nearrow 0$. Or equivalently, if $X_0 \subset X_1$, then $(X_1, X_0)_{\theta, p} \searrow X_0$ as $\theta \nearrow 1$ and $(X_1, X_0)_{\theta, p} \nearrow X_1$ as $\theta \searrow 1$. See Proposition 1.1.4 in [15] and section 1.3.3 in [21].

In the general case, one has the following equality but only for $\delta \in (0, \infty)$:

$$(11) \quad (E, D(A^m))_{\theta, q} = \left\{ x \in E : \int_0^\delta |t^{m(1-\theta)} (\omega_0 I + A)^m e^{-t(\omega_0 + A)} x|^p \frac{dt}{t} < \infty \right\}.$$

In this case, the formula (11) takes the following form

$$(12) \quad |x|_{D_{A^m}(\theta, q); \delta}^q = \int_0^\delta |t^{m(1-\theta)} A^m e^{-tA} x|^q \frac{dt}{t} + |x|^q.$$

Let us finally recall that if $0 < k < m \in \mathbb{N}$, $p \in [1, \infty]$ and $\theta \in (0, 1)$, then $(E, D(A^k))_{\theta, p} = (E, D(A^m))_{\frac{k}{m}\theta, p}$, see [21] Theorem 1.15.2 (f). Therefore, if $p \in [1, \infty)$ and $\theta \in [0, 1 - \frac{1}{p})$, then

$$(13) \quad D_A(\theta + \frac{1}{p}, q) = D_{A^2}(\frac{\theta}{2} + \frac{1}{2p}, q)$$

with equivalent norms.

Our main result in this note is the following

Theorem 2.1. *Under the above assumptions, for all $\theta \in (0, 1 - \frac{1}{p})$, there exists a constant $C = \hat{C}_\theta(E)$ such that for any process ξ described above and all $T \geq 0$, the following inequality holds*

$$(14) \quad \mathbb{E} \int_0^T |SC(\xi)(t)|_{D_A(\theta + \frac{1}{p}, q)}^p dt \leq C \mathbb{E} \int_0^T \int_S |\xi(t, z)|_{D_A(\theta, q)}^p \nu(dx) dt.$$

In the Gaussian case and $q = p = 2$, and E being a Hilbert space, the above result was proved by Da Prato in [7]. This result was then generalized to a class of so called Banach spaces of martingale type 2 in [1], see also [2], for nuclear Wiener process and in [4], to the case of cylindrical Wiener process. Finally, Da Prato and Lunardi studied in [8] the case when $p = 2$ and $q \geq 2$ for a one dimensional Wiener process. However, a generalisation of the last result to a cylindrical Wiener process does not cause any serious problems. We will state corresponding result at the end of this Note.

Theorem 2.1 will be deduced from a more general result whose idea can be traced back to Remark 1.

Theorem 2.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space, $p \in (1, 2]$ and $q \in [p, \infty)$. Let \mathcal{E}_p be a class of separable Banach spaces satisfying the following properties.*

(R1) With each space E belonging to the class \mathcal{E}_p we associate a separable Banach space $R = R(E)$ such that there is a family $(I_t)_{t \geq 0}$ of linear operators from the class $\mathcal{M}_{loc}^p(R(E))$ of all predictable $R(E)$ -valued processes to $L^p(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ such that for some constant $C = C_p > 0$

$$(15) \quad \mathbb{E}|I_t(\xi)|_E^p \leq C_p \mathbb{E} \left(\int_0^t \|\xi(r)\|_{R(E)}^p dr \right).$$

(R2) If $E \in \mathcal{E}_p$ and E_1 isomorphic to E , then E_1 belongs to \mathcal{E}_p as well.

(R3) If $E_1, E_2 \in \mathcal{E}_p$ and $\Phi : E_1 \rightarrow E_2$ is a bounded linear operator, then

$$\|\Phi \xi\|_{R(E_2)} \leq \|\Phi\| \|\xi\|_{R(E_1)}, \quad \xi \in R(E_1).$$

(R4) If (E_0, E_1) is an interpolation couple such that $E_1, E_2 \in \mathcal{E}_p$, then the real interpolation spaces $(E_0, E_1)_{\theta, p}$, $\theta \in (0, 1)$, belongs to \mathcal{E}_p as well.

(R5) For every $\delta > 0$ here exists a constant $K_\delta > 0$ such that

$$(16) \quad \int_0^\delta \|r^{1-\theta} A e^{-rA} \xi\|_{R(E)}^p \frac{dr}{r} \leq K_\delta^p \|\xi\|_{R(D_A(\theta, p))}^p, \quad \xi \in R(E).$$

(R6) There exists a constant $\hat{C}_q > 0$ such that for all $t > 0$

$$(17) \quad \mathbb{E}|I_t(\xi)|_E^q \leq \hat{C}_q \mathbb{E} \left(\int_0^t \|\xi(r)\|_{R(E)}^p dr \right)^{q/p}, \quad \xi \in \mathcal{M}_{loc}^p(R(E)).$$

Define another family $(SC_t)_{t \geq 0}$ of linear operators from $\mathcal{M}_{loc}^p(R(E))$ to $L^p(\Omega, \mathcal{F}_t, \mathbb{P}; E)$ by the following formula

$$(18) \quad SC_t(\xi) = I_t(e^{-(t-\cdot)A} \xi(\cdot)), \quad t \geq 0.$$

Then, for every $\theta \in (0, 1 - \frac{1}{p})$, there exists a constant $\hat{C}_{q, \theta}(E)$ such that for all $T > 0$ the following inequality holds

$$(19) \quad \mathbb{E} \int_0^T |SC_t(\xi)|_{D_A(\theta + \frac{1}{p}, q)}^q dt \leq \hat{C}_{q, \theta}(E) \mathbb{E} \int_0^T \|\xi(s)\|_{R(D_A(\theta + \frac{1}{p}, q))}^q dt.$$

Remark 2. It follows from (i) that if $\xi(r) = \eta(r)$ a.s. for a.a. $r \in [0, t]$, then $I_t(\xi) = I_t(\eta)$.

Now we shall present two basic examples.

Example 2.3. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space, $p = 2$. Let H be a separable Hilbert space and let \mathcal{E}_2 be a class of all 2-smoothable Banach spaces. With $E \in \mathcal{E}_2$ we associate the space $R(E) := R(H, E)$ of all γ -radonifying operators from H to E . It is

known, see [17] that $R(H, E)$ is a separable Banach space equipped with any of the following equivalent norms³, $2 \leq q < \infty$,

$$(20) \quad \|\varphi\|_{R(H,E);q}^q := \mathbb{E} \left| \sum_j \beta_j \varphi e_j \right|_E^q, \quad \varphi \in R(H, E),$$

$\{e_k\}_k$ be an ONB of H and $\{\beta_k\}_k$ a sequence of i.i.d. Gaussian $N(0,1)$ random variables.

Example 2.4. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space, $p \in (1, 2]$. Let (S, \mathcal{S}) be a measurable space and $\eta : \mathcal{S} \hat{\times} \mathcal{B}(\mathbb{R}_+) \rightarrow \mathbb{N}^+$ be a time homogeneous, compensated Poisson random measure over $(\Omega; \mathcal{F}; \mathbb{P})$ adapted to filtration $(\mathcal{F}_t)_{t \geq 0}$ with intensity $\nu \in M_S^+$. Let \mathcal{E}_p be the set of all separable Banach spaces of martingale type p . With $E \in \mathcal{E}_p$ we associated a measurable transformation $\xi : S \rightarrow E$ such that

$$\int_S |\xi(x)|_E^p \nu(dx) < \infty.$$

Then for $q \in [p, \infty)$ let

$$\|\xi\|_{R(E)}^q := \mathbb{E} \left| \int_0^1 \int_S \xi(x) \tilde{\eta}(dx, dr) \right|_E^q.$$

3. PROOF OF THEOREM 2.2

We begin with the case $q = p$. Without loss of generality the norm $|\cdot|_{D_A(\theta + \frac{1}{p}, p); 1}$, defined by formula (10), will be denoted by $|\cdot|_{D_A(\theta + \frac{1}{p}, p)}$. Also, we may assume that A^{-1} exists and is bounded so that the graph norm in $D(A)$ is equivalent to the norm $|A \cdot|$.

By the equality (13), definition (10), the Fubini Theorem and formula (18) we have

$$\begin{aligned} & \mathbb{E} \int_0^T |SC_t(\xi)|_{D_A(\theta + \frac{1}{p}, p)}^p dt \leq C \mathbb{E} \int_0^T |SC_t(\xi)|_{D_{A^2}(\frac{\theta}{2} + \frac{1}{2p}, p)}^p dt \\ &= C \int_0^T \int_0^1 \mathbb{E} |r^{2(1 - \frac{\theta}{2} - \frac{1}{2p})} A^2 e^{-rA} SC_t(\xi)|^p \frac{dr}{r} dt \\ &= C \int_0^T \int_0^1 r^{p(2-\theta)-1} \mathbb{E} |A^2 e^{-rA} I_t(e^{-(t-\cdot)A} \xi(\cdot))|^p \frac{dr}{r} dt \\ &= C \int_0^T \int_0^1 r^{p(2-\theta)-1} \mathbb{E} |I_t(A^2 e^{-rA} e^{-(t-\cdot)A} \xi(\cdot))|^p \frac{dr}{r} dt \leq \dots \end{aligned}$$

³Equivalence of the norms is a consequence of Khinchin-Kahane inequality.

By applying next the inequality (15), the property (R3), the Fubini Theorem, the fact that $|Ae^{-\frac{r}{2}A}| \leq Cr^{-1}$, $r > 0$, for some constant $C > 0$ as well as by observing that $\frac{1}{t-u+r} \leq \frac{1}{r}$ for $t \in [u, T]$, $r > 0$, we infer that

$$\begin{aligned}
\cdots &\leq C_p \int_0^1 r^{p(2-\theta)-1} \int_0^T \mathbb{E} \int_0^t \|A^2 e^{-(t-u+r)A} \xi(u)\|_{R(E)}^p du dt \frac{dr}{r} \\
&\leq C_p \int_0^1 r^{p(2-\theta)-1} \\
&\quad \int_0^T \mathbb{E} \int_0^t |Ae^{-\frac{t-u+r}{2}A}|^p \|Ae^{-\frac{t-u+r}{2}A} \xi(u)\|_{R(E)}^p du dt \frac{dr}{r} \\
&\leq C_p \mathbb{E} \int_0^1 r^{p(2-\theta)-1} \left[\sup_{0 \leq u \leq t} (t-u+r)^{-p} \right] \\
&\quad \int_0^T \left[\int_u^T \|Ae^{-\frac{t-u+r}{2}A} \xi(u)\|_{R(E)}^p dt \right] du \frac{dr}{r} \\
&\leq C_p \mathbb{E} \int_0^T \\
&\quad \int_0^{T+1-\rho} \|Ae^{-\frac{\sigma}{2}A} \xi(\rho)\|_{R(E)}^p \left[\int_{\rho \vee (\sigma+\rho-1)}^{\rho+\sigma} (\sigma+\rho-\tau)^{p(1-\theta)-2} d\tau \right] d\sigma d\rho \\
&\leq C_p \mathbb{E} \int_0^T \\
&\quad \int_0^{T+1-\rho} \|Ae^{-\frac{\sigma}{2}A} \xi(\rho)\|_{R(E)}^p \left[\int_\rho^{\rho+\sigma} (\sigma+\rho-\tau)^{p(1-\theta)-2} d\tau \right] d\sigma d\rho \\
&= C_p \mathbb{E} \int_0^T \int_0^{T+1-\rho} \|Ae^{-\frac{\sigma}{2}A} \xi(\rho)\|_{R(E)}^p \left[\int_0^\sigma \tau^{p(1-\theta)-2} d\tau \right] d\sigma d\rho \\
&= C'_p \mathbb{E} \int_0^T \int_0^{T+1-\rho} \sigma^{p(1-\theta)-1} \|Ae^{-\frac{\sigma}{2}A} \xi(\rho)\|_{R(E)}^p d\sigma d\rho \\
&\leq C''_p \mathbb{E} \int_0^T \int_0^{T/2} \|\sigma^{1-\theta} Ae^{-\sigma A} \xi(\rho)\|_{R(E)}^p \frac{d\sigma}{\sigma} d\rho \\
&\leq \hat{C}_p''' K_T^p \mathbb{E} \int_0^T \|\xi(r)\|_{R(D_A(\theta,p))}^p dr,
\end{aligned}$$

where the last inequality is a consequence of the assumption (R5).

The proof in the case $q > p$ follows the same ideas. Note also that the above prove resembles closely the proof from [8]. We give full details below.

We consider now the case $q > p$. We use the same notation as in the previous case. But we will make some (or the same) additional assumptions. By the equality (13), definition (10), the Fubini Theorem and formula (18) we have

$$\begin{aligned}
& \mathbb{E} \int_0^T |SC_t(\xi)|_{D_A(\theta+\frac{1}{p},q)}^q dt \leq C \mathbb{E} \int_0^T |SC_t(\xi)|_{D_{A^2}(\frac{\theta}{2}+\frac{1}{2p},q)}^q dt \\
& = C \int_0^T \int_0^1 s^{q(2-\theta)-\frac{q}{p}} \mathbb{E} |A^2 e^{-sA} SC_t(\xi)|^q \frac{ds}{s} dt \\
& = C \int_0^T \int_0^1 s^{q(2-\theta)-\frac{q}{p}} \mathbb{E} |A^2 e^{-sA} I_t(e^{-(t-\cdot)A} \xi(\cdot))|^q \frac{ds}{s} dt \\
& = C \int_0^T \int_0^1 s^{q(2-\theta)-\frac{q}{p}} \mathbb{E} |I_t(A^2 e^{-sA} e^{-(t-\cdot)A} \xi(\cdot))|^q \frac{ds}{s} dt \leq \dots
\end{aligned}$$

Before we continue, we formulate the following simple Lemma.

Lemma 3.1. *There exists a constant $C > 0$ such that for all $t > 0$, $s \in (0, 1)$*

$$\left(\int_0^t \frac{1}{(t-r+s)^{\frac{pq}{q-p}}} dr \right)^{\frac{q}{p}-1} \leq C \frac{1}{s^{q(1-\frac{1}{p})+1}}$$

Proof of Lemma 3.1. Denote $\alpha = \frac{pq}{q-p}$ and observe that $\alpha > 1$. Since $\int_0^t \frac{1}{(t-r+s)^\alpha} dr = \int_0^t \frac{1}{(r+s)^\alpha} dr \leq \int_0^\infty \frac{1}{(r+s)^\alpha} dr = \frac{1}{\alpha-1} \frac{1}{s^{\alpha-1}}$ and $(\alpha-1)(\frac{q}{p}-1) = q(1-\frac{1}{p})+1$, the result follows. \square

As in the earlier case, by applying the inequality (15), the property (R3), the Fubini Theorem, the fact that $|Ae^{-\frac{s}{2}A}| \leq Cs^{-1}$, $s > 0$, for some constant $C > 0$ as well as Hölder inequality and Lemma 3.1 we infer that

$$\begin{aligned}
\dots & \leq \hat{C}_q \int_0^1 s^{q(2-\theta)-\frac{q}{p}} \int_0^T \mathbb{E} \left[\int_0^t \|A^2 e^{-(t-r+s)A} \xi(r)\|_{R(E)}^p dr \right]^{q/p} dt \frac{ds}{s} \\
& \leq C \hat{C}_q \mathbb{E} \int_0^1 s^{q(2-\theta)-\frac{q}{p}} \int_0^T \left[\int_0^t |Ae^{-\frac{t-r+s}{2}A}|^p \|Ae^{-\frac{t-r+s}{2}A} \xi(r)\|_{R(E)}^p dr \right]^{q/p} dt \frac{ds}{s} \\
& \leq C \hat{C}_q \mathbb{E} \int_0^1 s^{q(2-\theta)-\frac{q}{p}} \int_0^T \left[\left(\int_0^t |Ae^{-\frac{t-r+s}{2}A}|^{\frac{pq}{q-p}} dr \right)^{\frac{q}{p}-1} \right. \\
& \quad \left. \int_0^t \|Ae^{-\frac{t-r+s}{2}A} \xi(r)\|_{R(E)}^q dr \right] dt \frac{ds}{s}
\end{aligned}$$

$$\begin{aligned}
&\leq C' \hat{C}_q \mathbb{E} \int_0^1 s^{q(2-\theta)-\frac{q}{p}} \int_0^T \frac{1}{s^{q(1-\frac{1}{p})+1}} \int_0^t \|Ae^{-\frac{t-r+s}{2}A} \xi(r)\|_{R(E)}^q dr dt \frac{ds}{s} \\
&= C' \hat{C}_q \mathbb{E} \int_0^1 s^{q(1-\theta)-1} \int_0^T \left[\int_r^T \|Ae^{-\frac{t-r+s}{2}A} \xi(r)\|_{R(E)}^q dt \right] dr \frac{ds}{s} \\
&\leq C' \hat{C}_q \mathbb{E} \int_0^T \int_0^{T+1-\rho} \|Ae^{-\frac{\sigma}{2}A} \xi(\rho)\|_{R(E)}^q \left[\int_{\rho \vee (\sigma+\rho-1)}^{\rho+\sigma} (\sigma + \rho - \tau)^{q(1-\theta)-2} d\tau \right] d\sigma d\rho \\
&\leq C' \hat{C}_q \mathbb{E} \int_0^T \int_0^{T+1-\rho} \|Ae^{-\frac{\sigma}{2}A} \xi(\rho)\|_{R(E)}^q \left[\int_\rho^{\rho+\sigma} (\sigma + \rho - \tau)^{q(1-\theta)-2} d\tau \right] d\sigma d\rho \\
&= C' \hat{C}_q \mathbb{E} \int_0^T \int_0^{T+1-\rho} \|Ae^{-\frac{\sigma}{2}A} \xi(\rho)\|_{R(E)}^q \left[\int_0^\sigma \tau^{q(1-\theta)-2} d\tau \right] d\sigma d\rho \\
&= \hat{C}_q' \mathbb{E} \int_0^T \int_0^{T+1-\rho} \sigma^{q(1-\theta)-1} \|Ae^{-\frac{\sigma}{2}A} \xi(\rho)\|_{R(E)}^q d\sigma d\rho \\
&\leq \hat{C}_q'' \mathbb{E} \int_0^T \int_0^{T/2} \|\sigma^{1-\theta} Ae^{-\sigma A} \xi(\rho)\|_{R(E)}^q \frac{d\sigma}{\sigma} d\rho \leq \\
&\leq \hat{C}_q''' K_{T/2}^p \mathbb{E} \int_0^T \|\xi(r)\|_{R(D_A(\theta, q))}^q dr
\end{aligned}$$

where the last inequality follows from Assumption R5. This completes the proof.

4. STOCHASTIC CONVOLUTION IN THE CYLINDRICAL GAUSSIAN CASE

Assume now that $W(t)$, $t \geq 0$, is a cylindrical Wiener process defined on some complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Let us denote by H the RKHS of that process, i.e. H is equal to the RKHS of $W(1)$.

Theorem 4.1. *Under the above assumptions there exists a constant $\hat{C}_q(E)$ such that for any process ξ described above the following inequality holds*

$$(21) \quad \mathbb{E} \int_0^T |SC(\xi)(t)|_{D_A(\theta+\frac{1}{p}, q)}^q dt \leq \hat{C}_q(E) \mathbb{E} \int_0^T \|\xi(t)\|_{R(H, D_A(\theta, q))}^q dt, \quad T \geq 0.$$

The proof of Theorem 4.1 will be preceded by the following useful result.

Proposition 4.2. *Let us assume that $\theta \in (0, 1)$, $q \geq 1$ and $T > 0$. Then there exists a constant $K_T > 0$ such that for each bounded linear map $\varphi : H \rightarrow E$ the following inequality holds*

$$\begin{aligned}
K_T^{-1} \|\varphi\|_{R(H, (E, D(A))_{\theta, q})}^q &\leq \int_0^T t^{(1-\theta)q} \|Ae^{-tA} \varphi\|_{R(H, E)}^q \frac{dt}{t} \\
(22) \qquad \qquad \qquad &\leq K_T \|\varphi\|_{R(H, (E, D(A))_{\theta, q})}^q.
\end{aligned}$$

In particular, $\varphi \in R(H, (D(A), E)_{\theta, q})$ iff (for some and/or all $T > 0$) the integral $\int_0^T t^{(1-\theta)q} \|Ae^{-tA} \varphi\|_{R(H, E)}^q \frac{dt}{t}$ is finite.

Proof of Proposition 4.2. Let $\{e_k\}_k$ be an ONB of H and $\{\beta_k\}_k$ a sequence of i.i.d. Gaussian $N(0,1)$ random variables. It is known, see e.g. [13] that there exists a constant $C_p(E)$ such that for each linear operator $\varphi : H \rightarrow E$ the following inequality holds.

$$(23) \quad C_p(E)^{-1} \mathbb{E} \left| \sum_j \beta_j \varphi e_j \right|_E^p \leq \|\varphi\|_{R(H, E)}^p \leq C_p(E) \mathbb{E} \left| \sum_j \beta_j \varphi e_j \right|_E^p$$

We have

$$\begin{aligned}
&\int_0^T t^{(1-\theta)q} \|Ae^{-tA} \varphi\|_{R(H, E)}^q \frac{dt}{t} \\
&\leq C_q(E) \int_0^T t^{(1-\theta)q} \mathbb{E} \left| \sum_k \beta_k Ae^{-tA} \varphi e_k \right|_E^q \frac{dt}{t} \\
&= C_q(E) \mathbb{E} \int_0^T t^{(1-\theta)q} \left| \sum_k \beta_k Ae^{-tA} \varphi e_k \right|_E^q \frac{dt}{t} \\
&= C_q(E) \mathbb{E} \left| \sum_k \beta_k Ae^{-tA} \varphi e_k \right|_{D_A(\theta, q); T}^q \\
&\leq C(T) C_q(E) \|\varphi\|_{R(H, D_A(\theta, q))}^q.
\end{aligned}$$

Since $D_A(\vartheta, q) = (E, D(A))_{\theta, q}$ with equivalent norms, this proves the second inequality in (22). The first inequality follows the same lines. \square

Proof of Theorem 4.1. From Proposition 4.2 we infer that the assumption (r5) in Theorem 2.2 is satisfied. Since it is well known that the other assumptions are also satisfied, see e.g. [3], the result follows from Theorem 2.2. \square

5. PROOF OF THEOREM 2.1

We only need to prove a version of Proposition 4.2 with $R(H, E)$ being replaced by $R(E) := L^p(S, \nu, E)$. We recall that here the measure space (S, \mathcal{S}, ν) is fixed for the whole section.

Proposition 5.1. *Let us assume that $\theta \in (0, 1)$, $q \geq 1$ and $T > 0$. Then there exists a constant $K_T > 0$ such that for each $\varphi \in L^p(S, \nu, E) =: R(E)$ the following inequality holds*

$$(24) \quad \begin{aligned} K_T^{-1} \|\varphi\|_{R((E, D(A))_{\theta, q})}^q &\leq \int_0^T t^{(1-\theta)q} \|Ae^{-tA} \varphi\|_{R(E)}^q \frac{dt}{t} \\ &\leq K_T \|\varphi\|_{R((E, D(A))_{\theta, q})}^q. \end{aligned}$$

In particular, $\varphi \in R((D(A), E)_{\theta, q})$ iff (for some and/or all $T > 0$) the integral $\int_0^T t^{(1-\theta)q} \|Ae^{-tA} \varphi\|_{R(E)}^q \frac{dt}{t}$ is finite.

Proof of Proposition 5.1. Follows by applying the Fubini Theorem. \square

APPENDIX A. MARTINGALE TYPE p , $p \in [1, 2]$, BANACH SPACES

In this section we collect some basic information about the martingale type p , $p \in [1, 2]$, Banach spaces.

Assume also that $p \in [1, 2]$ is fixed. A Banach space E is of martingale type p iff there exists a constant $L_p(E) > 0$ such that for all X -valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds

$$(25) \quad \sup_n \mathbb{E}|M_n|^p \leq L_p(E) \sum_{n=0}^N \mathbb{E}|M_n - M_{n-1}|^p,$$

where as usually, we put $M_{-1} = 0$.

Let us recall that a Banach space X is of type p iff there exists a constant $K_p(X) > 0$ for any finite sequence $\varepsilon_1, \dots, \varepsilon_n: \Omega \rightarrow \{-1, 1\}$ of symmetric i.i.d. random variables and for any finite sequence x_1, \dots, x_n of elements of X , the following inequality holds

$$(26) \quad \mathbb{E} \left| \sum_{i=1}^n \varepsilon_i x_i \right|^p \leq K_p(X) \sum_{i=1}^n |x_i|^p.$$

It is known, see e.g. [14, Theorem 3.5.2], that a Banach space X is of type p iff it is of Gaussian type p , i.e. there exists a constant $\tilde{K}_p(X) > 0$ such that for any finite sequence ξ_1, \dots, ξ_n of i.i.d. $N(0, 1)$ random variables and for any finite sequence x_1, \dots, x_n of elements of X , the following inequality holds

$$(27) \quad \mathbb{E} \left| \sum_{i=1}^n \xi_i x_i \right|^p \leq \tilde{K}_p(X) \sum_{i=1}^n |x_i|^p,$$

It is now well known, see e.g. Pisier [18] and [19], that X is of martingale type p iff it is p -smooth, i.e. there exists an equivalent norm $|\cdot|$

on X and there exist a constant $K > 0$ such that $\rho_X(t) \leq Kt^p$ for all $t > 0$, where $\rho_X(t)$ is the modulus of smoothness of $(X, |\cdot|)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2}(|x + ty| + |x - ty|) - 1 : |x|, |y| = 1 \right\}.$$

In particular, all spaces L^q for $q \geq p$ and $q > 1$, are of martingale type p .

Let us also recall that a Banach space X is an UMD space (i.e. X has the unconditional martingale difference property) iff for any $p \in (1, \infty)$ there exists a constant $\beta_p(X) > 0$ such that for any X -valued martingale difference $\{\xi_j\}$ (i.e.: $\sum_{j=1}^n \xi_j$ is a martingale), for any $\epsilon : \mathbb{N} \rightarrow \{-1, 1\}$ and for any $n \in \mathbb{N}$

$$(28) \quad \mathbb{E} \left| \sum_{j=1}^n \epsilon_j \xi_j \right|^p \leq \beta_p(X) \mathbb{E} \left| \sum_{j=1}^n \xi_j \right|^p.$$

It is known, see [5] and references therein, that for a Banach space X the following conditions are equivalent: i) X is an UMD space, (ii) X is ζ convex, (iii) the Hilbert transform for X -valued functions is bounded in $L^p(\mathbb{R}, X)$ for any (or some) $p > 1$.

Finally, it is known, see e.g. [18, Proposition 2.4], that if a Banach space X is both UMD and of type p , then X is of martingale type p .

APPENDIX B. PROOF OF INEQUALITY 5

In this appendix we formulate and prove inequality 5. Our approach is in a sense similar to the approach used in the Gaussian case by Neidhard [17] and Brzeźniak [2] or in the Poisson random measure in Madrekar and Rüdiger [16]. In fact, our main result below can be seen a generalisation of Theorem 3.6 from [16] to the case of martingale type p Banach spaces.

Notation 2. By $M_{S \times \mathbb{R}_+}^{\bar{\mathbb{N}}}$ we denote the family of all $\bar{\mathbb{N}}$ -valued measures on $(S \times \mathbb{R}_+, \mathcal{S} \otimes \mathcal{B}_{\mathbb{R}_+})$ and $\mathcal{M}_{S \times \mathbb{R}_+}^{\bar{\mathbb{N}}}$ is the σ -field on $M_{S \times \mathbb{R}_+}^{\bar{\mathbb{N}}}$ generated by functions $i_B : M \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{S} \otimes \mathcal{B}_{\mathbb{R}_+}$.

Let us assume that (S, \mathcal{S}) is a measurable space, $\nu \in M_S^+$ is a non-negative measure on (S, \mathcal{S}) and $\mathfrak{P} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space. We also assume that η is time homogeneous Poisson random measure over \mathfrak{P} , with the intensity measure ν , i.e. $\eta : (\Omega, \mathcal{F}) \rightarrow (M_S^{\bar{\mathbb{N}}}, \mathcal{M}_{S \times \mathbb{R}_+}^{\bar{\mathbb{N}}})$ is a measurable function satisfying the following conditions

- (i) for each $B \in \mathcal{S} \otimes \mathcal{B}_{\mathbb{R}_+}$, $\eta(B) := i_B \circ \eta : \Omega \rightarrow \bar{\mathbb{N}}$ is a Poisson random variable with parameter⁴ $\mathbb{E}\eta(B)$;
- (ii) η is independently scattered, i.e. if the sets $B_j \in \mathcal{S} \otimes \mathcal{B}_{\mathbb{R}_+}$, $j = 1, \dots, n$ are pair-wise disjoint, then the random variables $\eta(B_j)$, $j = 1, \dots, n$ are pair-wise independent;
- (iii) for all $B \in \mathcal{S}$ and $I \in \mathcal{B}_{\mathbb{R}_+}$, $\mathbb{E}[\eta(I \times B)] = \lambda(I)\nu(B)$, where λ is the Lebesgue measure;
- (iv) for each $U \in \mathcal{S}$, the $\bar{\mathbb{N}}$ -valued processes $(N(t, U))_{t \geq 0}$ defined by

$$N(t, U) := \eta((0, t] \times U), \quad t \geq 0$$

is $(\mathcal{F}_t)_{t \geq 0}$ -adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta((s, t] \times U)$ is independent of \mathcal{F}_s .

By $\tilde{\eta}$ we will denote the compensated Poisson random measure, i.e. a function defined by $\tilde{\eta}(B) = \eta(B) - \mathbb{E}(\eta(B))$, whenever the difference makes sense.

Lemma B.1. *Let $p \in (1, 2]$ and assume that E is a Banach space of martingale type p . If a finitely-valued function f belongs to $L^p(\Omega \times S, \mathcal{F}_a \otimes \mathcal{S}; \mathbb{P} \otimes \nu; E)$ for some $a \in \mathbb{R}_+$, then for any $b > a$,*

$$(29) \quad \mathbb{E} \left| \int_S f(x) \tilde{\eta}(dx, (a, b]) \right|_E^p \leq 2^{2-p} L_p(E) (b-a) \mathbb{E} \int_S |f(x)|_E^p \nu(dx)$$

Since the space of finitely-valued functions is dense in $L^p(\Omega \times S, \mathcal{F}_a \otimes \mathcal{S}; \mathbb{P} \otimes \nu; E)$, see e.g. Lemma 1.2.14 in [6].

Corollary B.2. *Under the assumptions of Lemma B.1 there exists a unique bounded linear operator*

$$\tilde{I}_{(a,b)} : L^p(\Omega \times S, \mathcal{F}_a \otimes \mathcal{S}; \mathbb{P} \otimes \nu; E) \rightarrow L^p(\Omega, \mathcal{F}, E)$$

such that for a finitely-valued function f , we have

$$\tilde{I}_{(a,b)}(f) = \int_S f(x) \tilde{\eta}(dx, (a, b]).$$

In particular, for every $f \in L^p(\Omega \times S, \mathcal{F}_a \otimes \mathcal{S}; \mathbb{P} \otimes \nu; E)$,

$$(30) \quad \mathbb{E} |\tilde{I}_{(a,b)}(f)|_E^p \leq 2^{2-p} L_p(E) (b-a) \mathbb{E} \int_S |\xi(x)|_E^p \nu(dx).$$

In what follows, unless we in danger of ambiguity, for every $L^p(\Omega \times S, \mathcal{F}_a \otimes \mathcal{S}; \mathbb{P} \otimes \nu; E)$ we will write $\int_S \xi(x) \tilde{\eta}(dx, (a, b])$ instead of $\tilde{I}_{(a,b)}(f)$.

⁴If $\mathbb{E}\eta(B) = \infty$, then obviously $\eta(B) = \infty$ a.s..

Let X be any Banach space. Later on we will take X to be one of the spaces E , $R(H, E)$ or $L^p(S, \nu, E)$. For $a < b \in [0, \infty]$ let $\mathcal{N}(a, b; X)$ be the space of (equivalence classes of) predictable functions $\xi : (a, b] \times \Omega \rightarrow X$.

For $q \in (1, \infty)$ we set

$$(31) \quad \mathcal{N}^q(a, b; X) = \left\{ \xi \in \mathcal{N}(a, b; X) : \int_a^b |\xi(t)|^q dt < \infty \text{ a.s.} \right\},$$

$$(32) \quad \mathcal{M}^q(a, b; X) = \left\{ \xi \in \mathcal{N}(a, b; X) : \mathbb{E} \int_a^b |\xi(t)|^q dt < \infty \right\}.$$

Let $\mathcal{N}_{\text{step}}(a, b; X)$ be the space of all $\xi \in \mathcal{N}(a, b; X)$ for which there exists a partition $a = t_0 < t_1 < \dots < t_n < b$ such that for $k \in \{1, \dots, n\}$, for $t \in (t_{k-1}, t_k]$, $\xi(t) = \xi(t_k)$ is $\mathcal{F}_{t_{k-1}}$ -measurable and $\xi(t) = 0$ for $t \in (t_n, b)$. We put $\mathcal{M}_{\text{step}}^q = \mathcal{M}^q \cap \mathcal{N}_{\text{step}}$. Note that $\mathcal{M}^q(a, b; X)$ is a closed subspace of $L^q([a, b] \times \Omega; X) \cong L^q([a, b]; L^q(\Omega; X))$.

In what follows we put $a = 0$ and $b = \infty$. For $\xi \in \mathcal{M}_{\text{step}}^p(0, \infty; L^p(S, \nu; E))$ we set

$$(33) \quad \tilde{I}(\xi) = \sum_{j=1}^n \int_S \xi(t_j, x) \tilde{\eta}(dx, (t_{j-1}, t_j]).$$

Obviously, $\tilde{I}(\xi)$ is a \mathcal{F} -measurable map from Ω with values in E .

We have the following auxiliary results.

Lemma B.3. *Let $p \in (1, 2]$ and assume that E is a Banach space of martingale type p . Then for any $\xi \in \mathcal{M}_{\text{step}}^p(0, \infty; L^p(S, \nu; E))$, $\tilde{I}(\xi) \in L^p(\Omega, E)$, $\mathbb{E}\tilde{I}(\xi) = 0$ and*

$$(34) \quad \mathbb{E}|\tilde{I}(\xi)|^p \leq L_p^2(E) 2^{2-p} \int_0^\infty \mathbb{E} \int_S |\xi(t, x)|_E^p \nu(dx) dt$$

Lemma B.4. *Suppose that $\xi \sim \text{Pois}(\lambda)$, where $\lambda > 0$. Then, for all $p \in [1, 2]$,*

$$\mathbb{E}|\xi - \lambda|^p \leq 2^{2-p} \lambda.$$

Remark 3. *One can easily calculate that*

$$\mathbb{E}(|\xi - \lambda|) = 2\lambda e^{-\lambda}, \quad \text{if } \lambda \in (0, 1).$$

Theorem B.5. *Assume that $p \in (1, 2]$ and E is a martingale type p Banach space. Then there exists there exists a unique bounded linear operator*

$$\tilde{I} : \mathcal{M}^p(0, \infty, L^p(S, \nu; E)) \rightarrow L^p(\Omega, \mathcal{F}, E)$$

such that for $\xi \in \mathcal{M}_{\text{step}}^p(0, \infty, L^p(S, \nu; E))$ we have $I(\xi) = \tilde{I}(\xi)$. In particular, for every $\xi \in \mathcal{M}^p(0, \infty, L^p(S, \nu; E))$,

$$(35) \quad \mathbb{E}|I(\xi)|_E^p \leq 2^{2-p} L_p^2(E) \mathbb{E} \int_0^\infty \int_S |\xi(t, x)|_E^p \nu(dx) dt.$$

Proof of Theorem B.5. Follows from Lemma B.3 and the density of $\mathcal{M}_{\text{step}}^p(0, \infty, L^p(S, \nu; E))$ in the space $\mathcal{M}^p(0, \infty, L^p(S, \nu; E))$. \square

In a natural way we can define spaces $\mathcal{M}_{\text{loc}}^p(0, \infty, L^p(S, \nu; E))$ and $\mathcal{M}^p(0, T, L^p(S, \nu; E))$, where $T > 0$. Then for any $\xi \in \mathcal{M}_{\text{loc}}^p(0, \infty, L^p(S, \nu; E))$ we can in a standard way define the integral $\int_0^t \int_S \xi(r, x) \tilde{\eta}(dx, dr)$, $t \geq 0$, as the càdlàg modification of the process

$$(36) \quad I(1_{[0, t]} \xi), \quad t \geq 0,$$

where $[1_{[0, t]} \xi](r, x; \omega) := 1_{[0, t]}(r) \xi(r, x, \omega)$, $t \geq 0$, $r \in \mathbb{R}_+$, $x \in S$ and $\omega \in \Omega$. To show that this càdlàg modification exists we argue as follows. First of all we can assume that $\mathcal{M}^p(0, T, L^p(S, \nu; E))$, for some $T > 0$. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be an $\mathcal{M}_{\text{step}}^p(0, T, L^p(S, \nu; E))$ -valued sequence that is convergent in $\mathcal{M}^p(0, T, L^p(S, \nu; E))$ to ξ . Hence, the sequence $\{\xi_n, n \in \mathbb{N}\}$ is uniformly integrable and so it follows that the condition (a) in Remark 3.8.7 from [10] is satisfied. Similarly, the compact containment condition, i.e. the condition (a) in Theorem 3.7.2 from [10], holds true in view of the Prohorov Theorem, since for any $t \geq 0$ the laws of the sequence $\{I(1_{[0, t]} \xi_n), n \in \mathbb{N}\}$ are tight in the set of all probability measures over E , compare also with [9].

Similarly, for a stopping time τ we can define and process $\xi \in \mathcal{M}_{\text{loc}}^p(0, \infty, L^p(S, \nu; E))$ and the integral

$$(37) \quad \int_0^\tau \int_S \xi(r, x) \tilde{\eta}(dx, dr) := I(1_{[0, \tau]} \xi),$$

provided $1_{[0, \tau]} \xi \in \mathcal{M}^p(0, \infty, L^p(S, \nu; E))$. Theorem B.5 implies that in this case the following inequality holds.

$$(38) \quad \mathbb{E} \left| \int_0^\tau \int_S \xi(r, x) \tilde{\eta}(dx, dr) \right|_E^p \leq C_p \mathbb{E} \int_0^\tau \int_S |\xi(r, x)|_E^p \nu(dx) dr.$$

with some constant $C_p > 0$ independent of ξ .

Proof of Lemma B.3. Let us observe that the sequence $(M_k)_{k=1}^n$ defined by $M_k = \sum_{j=1}^k \int_S \xi(t_j, x) \tilde{\eta}(dx, [t_{j-1}, t_j])$ is an E -valued martingale (with respect to the filtration $(\mathcal{F}_{t_k})_{k=1}^n$). Therefore, by the martingale type p property of the space E and Lemma B.1 we have the

following sequence of inequalities

$$\begin{aligned}
 \mathbb{E}|\tilde{I}(\xi)|_E^p &= \mathbb{E}|M_n|_E^p \leq L_p(E) \sum_{k=1}^n \mathbb{E} \left| \int_S \xi(t_k, x) \tilde{\eta}(dx, [t_{k-1}, t_k]) \right|_E^p \\
 (39) \quad &\leq L_p^2(E) 2^{2-p} \sum_{k=1}^n (t_k - t_{k-1}) \mathbb{E} \int_S |\xi(t_k, x)|_E^p \nu(dx) \\
 &= L_p^2(E) 2^{2-p} \int_0^\infty \mathbb{E} \int_S |\xi(t, x)|_E^p \nu(dx) dt.
 \end{aligned}$$

This concludes the proof. \square

Proof of Lemma B.1. Put $I = (a, b]$. We may suppose that $f = \sum_i f_i 1_{A_i \times B_i}$ with $f_i \in E$, $A_i \in \mathcal{F}_a$ and $B_i \in \mathcal{S}$, the finite family of sets $(A_i \times B_i)$ being pair-wise disjoint and $\nu(B_i) < \infty$. Let us notice that

$$\int_S f(x) \tilde{\eta}(dx, I) = \sum_i 1_{A_i} \tilde{\eta}(B_i \times I) f_i.$$

Since the random variables $\tilde{\eta}(B_i \times I)$ are independent from the σ -field \mathcal{F}_a , the random variables $1_{A_i} \tilde{\eta}(B_i \times I)$ conditioned on \mathcal{F}_a are independent and so by the martingale type p property of the space E and Lemma B.1 we infer that

$$\begin{aligned}
 \mathbb{E} \left| \int_S \xi(x) \tilde{\eta}(dx, I) \right|_E^p &= \mathbb{E} \left[\mathbb{E} \left(\left| \sum_i 1_{A_i} \tilde{\eta}(B_i \times I) f_i \right|_E^p \middle| \mathcal{F}_a \right) \right] \\
 &\leq \mathbb{E} \left[L_p(E) \sum_i |f_i|_E^p \mathbb{E} |\tilde{\eta}(B_i \times I)|^p \right] \\
 &\leq L_p(E) \mathbb{E} \left[\sum_i |f_i|_E^p 1_{A_i} 2^{2-p} \lambda(I) \nu(B_i) \right] \\
 &= 2^{2-p} L_p(E) \sum_i |f_i|_E^p \nu(B_i) \lambda(I) \mathbb{P}(A_i) \\
 &= 2^{2-p} L_p(E) \lambda(I) \int_{\Omega \times S} \left| \sum_i f_i 1_{A_i \times B_i} \right|^p d(\mathbb{P} \otimes \nu) \\
 &= \tilde{L}_p(E) (b - a) \mathbb{E} \int_S |f(x)|_E^p \nu(dx).
 \end{aligned}$$

The proof is complete. \square

Proof of Lemma B.4. The case $p = 2$ is well known. Since $\xi \geq 0$ and $\mathbb{E}(\xi) = \lambda$, the case $p = 1$ follows by the triangle inequality. The case $p \in (1, 2)$ follows then by applying the Hölder inequality. Indeed, with $\alpha = 2(p - 1)$ and $\beta = 2 - p$ we have the following sequence of

inequalities, where $\eta := |\xi - \lambda|$.

$$\begin{aligned}\mathbb{E}(\eta^p) &= \mathbb{E}(\eta^\alpha \eta^\beta) \leq [\mathbb{E}((\eta^\alpha)^{2/\alpha})]^{\alpha/2} [\mathbb{E}((\eta^\beta)^{1/\beta})]^\beta \\ &= [\mathbb{E}(\eta^2)]^{\alpha/2} [\mathbb{E}(\eta)]^\beta \leq (\lambda)^{\alpha/2} (2\lambda)^\beta = 2^{2-p} \lambda.\end{aligned}$$

□

We conclude with a result corresponding to inequality (6).

Corollary B.6. *Assume that $1 < q \leq p < 2$ and E is a martingale type p Banach space. Then there exists there exists a constant $C > 0$ such that for any process $\xi \in \mathcal{M}_{\text{loc}}^p(0, \infty, L^p(S, \nu; E)) \rightarrow L^p(\Omega, \mathcal{F}, E)$, and any $T > 0$,*

(40)

$$\mathbb{E} \left| \sup_{t \in [0, T]} \int_0^t \int_S \xi(r, x) \tilde{\eta}(dx, dr) \right|^q \leq C \mathbb{E} \left(\int_0^T \int_S |\xi(r, x)|^p \nu(dx) dr \right)^{q/p}.$$

The proof of the above result will be based on Proposition IV.4.7 from the monograph B.7 by Revuz and Yor which we recall here for the convenience of the reader.

Proposition B.7. *Suppose that a positive, adapted right-continuous process Z is dominated by an increasing process A , with A_0 , i.e. there exists a constant $C > 0$ such that for every bounded stopping time τ , $\mathbb{E}Z_\tau \leq C\mathbb{E}A_\tau$. Then for any $k \in (0, 1)$,*

$$\mathbb{E} \sup_{0 \leq t < \infty} Z_t^k \leq C^k \frac{2-k}{1-k} \mathbb{E}A_\infty^k.$$

Proof of Corollary B.6. Let now fix $q \in (1, p)$. Put $k = q/p$. We will apply Proposition B.7 to the processes $Z_t = \left| \int_0^t \int_S \xi(r, x) \tilde{\eta}(dx, dr) \right|_E^p$ and $A_t = \int_0^t \int_S |\xi(r, x)|_E^p \nu(dx) dr$, $t \in [0, T]$. Let us notice that in view of inequality (38), the process Z is dominated by the process A . Since Z is right continuous, $\sup_{0 \leq t \leq T} Z_t^k = \sup_{0 \leq t \leq T} \left| \int_0^t \int_S \xi(r, x) \tilde{\eta}(dx, dr) \right|_E^q$ and $A_\infty^k = \left(\int_0^T \int_S |\xi(r, x)|_E^p \nu(dx) dr \right)^{q/p}$, we get inequality (40). This completes the proof of Corollary B.6. □

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